A Duality Theorem for Iterated Infinite Cyclic Coverings (多様体に於ける低次元トポロジーの問題)

KAWAUCHI, AKIO

数理解析研究所講究録
309巻 1977年 124-145

A duality theorem for iterated infinite cyclic coverings

by

Akio Kawauchi

This paper will derive a dual structure immanent in a manifold by establishing a duality analogous to the classical Poincaré duality for iterated infinite cyclic covers of a compact manifold. An application will be done in Section three. Spaces and maps will be considered in the piecewise-linear category, unless otherwise stated.

1. Preliminaries

Let X^n be a compact, connected, piecewise-linear n-manifold and suppose we are given a sequence $X^{(N)} \to X^{(N-1)} \to \cdots \to X^{(2)} \to X^{(1)} \to X^{(0)} = X^n$ such that for each i, $i = 1,2,..,N-1$, $X^{(i+1)} \to X^{(i)}$ is the composite of $X^{(i+1)} \to \hat{X}^{(i)} \to X^{(i)}$, where $\hat{X}^{(i)} \to X^{(i)}$ is a finitely sheeted (possibly irregular) connected covering and $X^{(i+1)} \to \hat{X}^{(i)}$ is an infinite cyclic connected covering (,that is, a regular connected covering whose covering translation group is infinite cyclic). We use the notation $\hat{Y}$ for a finitely sheeted connected cover of a space Y throughout the paper.

1.1 Definition. For $k \leq 0$ $f(k)$ is the class of arbitrary spaces. $f(1)$ is the class of connected spaces with finitely generated fundamental groups. For $k \geq 2$, $f(k)$ is the class of connected spaces Y with $\pi_1(Y)$ finitely generated and such that the integral group ring $Z[\pi_1(Y)]$ is Noetherian and $H_i(\overline{Y};Z)$ is a finitely generated left $Z[\pi_1(Y)]$-module for $i \leq k-1$ and $H_k(\overline{Y};Q)$ is a finitely generated left $Q[\pi_1(Y)]$-module. (Throughout the paper, $\overline{Y}$ **denotes the** universal cover of a space Y.)

1.2 Lemma. $\pi_1(X^{(N)})$ is finitely presented, if it is finitely generated.

This follows rapidly from **Lemma 1.3 below** and the induction on N [by considering $X^{(N)} \times S^m$ for a large m, if possible].

1.3 Lemma. Let $n \geq 5$. If $\pi_1(X^{(1)})$ is finitely generated, then there is a map $\varphi: X \to S^1$ such that for a point $p \in S^1$ $\varphi^{-1}(p)$ is a connected compact (piecewise-linear, proper bicollared) submanifold of $X^{(1)}$ such that the natural homomorphism $\pi_1(\varphi^{-1}(p)) \to \pi_1(X^{(1)})$ is an isomorphism.

Proof. Let $X^{(1)}$ is obtained from a simplicial map $\varphi_1: X \to S^1$. For a non-vertex point $p \in S^1$ $\varphi_1^{-1}(p)$ is a compact submanifold of $X^{(1)}$. Since $\pi_1(X^{(1)})$ is finitely generated, from an argument of J. Stallings[12] we can assume that $\varphi_1^{-1}(p)$ is connected. Next, since $n \geq 5$, we can kill the kernel of $\pi_1(\varphi_1^{-1}(p)) \to \pi_1(X^{(1)})$ by a surgery by an argument of M.A.

Gutiérrez[2]. Thus we have a map $\varphi : X \to S^1$ homotopic to φ_1 such that $\varphi^{-1}(p)$ is connected compact submanifold of $X^{(1)}$ and $\pi_1(\varphi^{-1}(p)) \to \pi_1(X^{(1)})$ is a monomorphism. An argument of L.P.Neuwirth[8], then, implies that this monomorphism must be an isomorphism. This completes the proof.

By Lemma 1.2, if $X^{(N)} \in f(1)$, then $\pi_1(X^{(N)})$ is finitely presented. So, one may note that the class $f(k)$ is closely related to the finiteness condition studied in detail by C.T.C.Wall[13] at least for manifolds of the type of $X^{(N)}$.

It seems difficult to know whether or not the integral group ring of a group is Noetherian. A partial result on this is as follows:

1.4 Lemma. Let G_o be a subgroup of a group G with finite index or with infinite cyclic quotient group. If $Z[G_o]$ is left Noetherian, then $Z[G]$ is left Noetherian.

Proof. If G/G_o has a finite index, $Z[G]$ can be considered as a finitely generated left module over $Z[G_o]$. Since $Z[G_o]$ is left Noetherian, it follows immediately that $Z[G]$ is left Noetherian. If G/G_o is infinite cyclic, then $Z[G]$ can be considered as a polynomial ring with negative exponents and with right coefficients in $Z[G_o]$. Then it follows that $Z[G]$ is left Noetherian by using the proof of the Hilbert basis theorem.

2. A duality theorem

We will state our duality theorem individually on the iterated number N of infinite cyclic coverings.

Duality Theorem 0(Poincaré Duality). Suppose $\hat{X}^{(0)}$ is orientable. There is a duality

$$\cap\hat{\mu}^{(0)}: H^i(\hat{X}^{(0)},\partial\hat{X}^{(0)};Z) \approx H_{n-i}(\hat{X}^{(0)};Z) \quad \text{for all} \quad i.$$

This is widely known, since $\hat{X}^{(0)}$ is compact.

Duality Theorem 1. Suppose $\hat{X}^{(1)}$ is orientable. If $H_i(\hat{X}^{(1)},\partial\hat{X}^{(1)};Z)$ is finitely generated abelian for $i \leq m$ and $\dim_Q H_{m+1}(\hat{X}^{(1)},\partial\hat{X}^{(1)};Q) < +\infty$, then there is a duality

$$\cap\hat{\mu}^{(1)}: H^i(\hat{X}^{(1)},\partial\hat{X}^{(1)};Z) \approx H_{n-1-i}(\hat{X}^{(1)};Z)$$

for all $i \leq m$ and for $i = m+1$ this map is a monomorphism.

This is a simple version of a known result. (See[4].)

An outline of the proof is as follows: Let $N_p^+ \supset N_{p+1}^+ \supset \cdots$ and $N_q^- \supset N_{q+1}^- \supset \cdots$ be the neighborhoods of the two ends of $X^{(1)}$ as in [4] or [6]. Let $\hat{N}_p^+ \supset \hat{N}_{p+1}^+ \supset \cdots$ and $\hat{N}_q^- \supset \hat{N}_{q+1}^- \supset \cdots$ be the lifts, which are still the neighborhoods of ends of $\hat{X}^{(1)}$, since $\hat{X}^{(1)}$ has still two ends. (See D.B.A.Epstein[1].) Using these neighborhoods, from an analogous method of [4] or [6] we obtain that

$$\cap\hat{\mu}^{(1)}: H^i(\hat{X}^{(1)},\partial\hat{X}^{(1)};Z) \xrightarrow{\delta} \lim_{p,q\to\infty} H^{i+1}(\hat{X}^{(1)},\partial\hat{X}^{(1)}\cup\hat{N}_p^+\cup\hat{N}_q^-;Z)$$

$= H_c^{i+1}(\hat{X}^{(1)},\partial\hat{X}^{(1)};Z) \overset{\cap[\hat{X}^{(1)}]}{\approx} H_{n-1-i}(\hat{X}^{(1)};Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. This completes the outlined proof.

<u>Duality Theorem 2</u>. <u>Suppose</u> $\hat{X}^{(2)}$ <u>is orientable and</u> $X^{(1)} \in f(m+2)$ <u>and</u> $\partial X^{(1)}$ <u>has at most finitely many components each of which is in</u> $f(m+1)$. <u>If</u> $H_i(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z)$ <u>is finitely generated abelian for</u> $i \leq m$ <u>and</u> $\dim_Q H_{m+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)};Q) < +\infty$, <u>then there is a duality</u>

$$\cap\hat{\mu}^{(2)}: H^i(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z) \approx H_{n-2-i}(\hat{X}^{(2)};Z)$$

<u>for all</u> $i \leq m$ <u>and for</u> $i = m+1$ <u>this map is a monomorphism</u>.

To prove Duality Theorem 2, we use the following lemma:

2.1 <u>Lemma</u>. <u>Let</u> $n \geq 6$ <u>and</u> $1 \leq k \leq (n-3)/2$. <u>Suppose</u> $X^{(1)} \in f(k)$ <u>and</u> $\partial X^{(1)}$ <u>has at most finitely many components each of which is in</u> $f(k-1)$. <u>Then there is a proper map</u> $\varphi: X^{(1)} \to R^1$ <u>such that for a point</u> $p \in R^1$ $\varphi^{-1}(p) = M$ <u>is a compact connected (piecewise-linear proper bicollared) submanifold with</u> $\partial X^{(1)} \cup M$ <u>connected and such that</u> $\pi_1(M) \overset{\approx}{\to} \pi_1(X^{(1)})$ <u>and</u> $\pi_i(\partial M) \overset{\approx}{\to} \pi_i(\partial X^{(1)})$ $i = 0,1$ <u>(if</u> $k \geq 2$) <u>and such that for any cover</u> $(\tilde{X}^{(1)},\partial\tilde{X}^{(1)}\cup\tilde{M})$ <u>of</u> $(X^{(1)},\partial X^{(1)}\cup M)$ $H_i(\tilde{X}^{(1)},\partial\tilde{X}^{(1)}\cup\tilde{M}) = 0$ <u>with integral coefficients for</u> $i \leq k-1$ <u>or</u> $i \leq k = 1$ <u>and with rational coefficients for</u> $i = k \geq 2$. <u>Furthermore, we can have</u> $H_i(\tilde{M}) \overset{\approx}{\to} H_i(\tilde{X}^{(1)})$ <u>with integral coefficients for</u> $i \leq k-1$ <u>or</u> $i \leq k = 1$ <u>and with rational coefficients for</u> $i = k \geq 2$. <u>For</u> $k \geq 2$, $H_i(\partial\tilde{M}) \overset{\approx}{\to} H_i(\partial\tilde{X}^{(1)})$

<u>with integral coefficients for</u> $i \leq k-2$ <u>or</u> $i \leq k-1=1$ <u>and with rational coefficients for</u> $i = k-1 \geq 2$.

Proof. By Lemma 1.3 there is a map $\varphi_1 : X \to S^1$ such that $\varphi_1^{-1}(p_1)$ is a compact connected submanifold of $X^{(1)}$ with $\partial X^{(1)} \cup \varphi_1^{-1}(p_1)$ connected and $\pi_1(\varphi_1^{-1}(p_1)) \approx \pi_1(X^{(1)})$ and $\pi_i(\partial\varphi_1^{-1}(p_1)) \approx \pi_i(\partial X^{(1)})$, $i = 0,1$ (if $k \geq 2$). [Since $\partial X^{(1)}$ has only finite components, each components of $\partial X^{(1)}$ must intersect with $\varphi_1^{-1}(p_1)$. For $k \geq 2$ first apply Lemma 1.3 for each component of $\partial X^{(1)}$.] Let $\varphi : X^{(1)} \to \mathbb{R}^1$ be a lift of φ_1. Note that $\varphi^{-1}(p) = \varphi_1^{-1}(p_1)$ for a lift $p \in \mathbb{R}^1$ of p_1. The rest of the proof follows from Lemma 2.2 below and the following simple assertion: If $H_i(\tilde{X}^{(1)}, \tilde{M}) = 0$ and $H_{i-1}(\partial\tilde{X}^{(1)}, \partial\tilde{M}) = 0$, then $H_i(\tilde{X}^{(1)}, \partial\tilde{X}^{(1)} \cup \tilde{M}) = 0$. This follows from the homology exact sequence of the triple $\tilde{X}^{(1)} \supset \partial\tilde{X}^{(1)} \cup \tilde{M} \supset \tilde{M}$. In fact,

$$\begin{array}{ccccc}
H_i(\tilde{X}^{(1)}, \tilde{M}) & \to & H_i(\tilde{X}^{(1)}, \partial\tilde{X}^{(1)} \cup \tilde{M}) & \to & H_{i-1}(\partial\tilde{X}^{(1)} \cup \tilde{M}, \tilde{M}) \\
\| & & & & \| \\
0 & & & & H_{i-1}(\partial\tilde{X}^{(1)}, \partial\tilde{M}) \\
 & & & & \| \\
 & & & & 0
\end{array}$$

Hence $H_i(\tilde{X}^{(1)}, \partial\tilde{X}^{(1)} \cup \tilde{M}) = 0$.

2.2 <u>Lemma</u>. <u>Let</u> $2 \leq k \leq (n-3)/2$ <u>and</u> $X^{(1)} \in f(k)$. <u>Assume</u> $\pi_1(\varphi^{-1}(p)) \approx \pi_1(X^{(1)})$ <u>for a proper map</u> $\varphi : X^{(1)} \to \mathbb{R}^1$ <u>with</u> $\varphi^{-1}(p)$ <u>a compact connected submanifold. Then there is a proper map</u> $\varphi' : X^{(1)} \to \mathbb{R}^1$ <u>homotopic to</u> φ <u>by a homotopy with compact support such that</u> $\varphi'^{-1}(p)$ <u>is a compact connected submanifold and</u> $\pi_1(\varphi'^{-1}(p)) \approx \pi_1(X^{(1)})$ <u>and</u> $H_i(\tilde{X}^{(1)}, \tilde{\varphi}'^{-1}(p)) = 0$ <u>with integral coefficients for</u> $i \leq k-1$ <u>and rational coefficients for</u> $i = k$.

<u>Moreover we can have</u> $H_i(\widetilde{\varphi}'^{-1}(p)) \approx H_i(\widetilde{X}^{(1)})$ <u>with integral coefficients for</u> $i \leq k-1$ <u>and rational coefficients for</u> $i = k$.

Proof. $H_i(\widetilde{X}^{(1)}, \widetilde{\varphi}'^{-1}(p)) = 0$ follows from a surgerical argument of L.C.Siebenmann[10]. [First, note that $H_2(\overline{X}^{(1)}, \overline{\varphi}^{-1}(p)) = H_2(\overline{X}_+, \overline{\varphi}^{-1}(p)) + H_2(\overline{X}_-, \overline{\varphi}^{-1}(p)) = \pi_2(X_+, \varphi^{-1}(p)) + \pi_2(X_-, \varphi^{-1}(p))$ is a finitely generated $Z[\pi_1(X^{(1)})]$-module (or $Q[\pi_1(X^{(1)})]$-module for $k = 2$), where $\overline{X}^{(1)} = \overline{X}_+ \cup \overline{X}_-$ with $\overline{X}_+ \cap \overline{X}_- = \overline{\varphi}^{-1}(p)$. Hence by killing the generators, we may have a proper map $\varphi_1 : X^{(1)} \to \mathbb{R}^1$ homotopic to φ by a homotopy with compact support such that $\varphi_1^{-1}(p)$ is a compact connected submanifold and $\pi_1(\varphi_1^{-1}(p)) \approx \pi_1(X^{(1)})$ and $H_2(\overline{X}^{(1)}, \overline{\varphi}_1^{-1}(p)) = 0$. Similar for $i \geq 2$.] Next, note that $\pi_i(X^{(1)}, \varphi'^{-1}(p)) = 0$ with Z **coefficients** for $i \leq k-1$ and Q coefficients for $i=k$, and $\pi_{k+1}(X^{(1)}, \varphi'^{-1}(p)) = H_{k+1}(\overline{X}^{(1)}, \overline{\varphi}'^{-1}(p))$ with Q coefficients by the relative Hurewicz isomorphism theorem (modulo torsion). Consider the exact sequence of the following part: $H_k(\overline{X}^{(1)}, \overline{\varphi}'^{-1}(p)) \to H_{k-1}(\overline{\varphi}'^{-1}(p)) \xrightarrow{i_*} H_{k-1}(\overline{X}^{(1)}) \to 0$. Since $H_{k-1}(\overline{\varphi}'^{-1}(p))$ is finitely generated over $Z[\pi_1(X^{(1)})]$ and $Z[\pi_1(X^{(1)})]$ is Noetherian, we obtain that Ker i_* is finitely generated over $Z[\pi_1(X^{(1)})]$. Since $H_k(\overline{X}^{(1)}, \overline{\varphi}'^{-1}(p)) = H_k(\overline{X}_+, \overline{\varphi}'^{-1}(p)) + H_k(\overline{X}_-, \overline{\varphi}'^{-1}(p)) = \pi_k(X_+, \varphi'^{-1}(p)) + \pi_k(X_-, \varphi'^{-1}(p))$, we can kill the generators of Ker i_* by a surgery and hence we can assume that the map $\pi_k(X^{(1)}, \varphi'^{-1}(p)) = H_k(\overline{X}^{(1)}, \overline{\varphi}'^{-1}(p)) \to H_{k-1}(\overline{\varphi}'^{-1}(p))$ with Z coefficients is a trivial homomorphism. This implies that $i_* : H_{k-1}(\widetilde{\varphi}'^{-1}(p); Z) \approx H_{k-1}(\widetilde{X}^{(1)}; Z)$. With Q coefficients, the same argument is applicable for the following part: $H_{k+1}(\overline{X}^{(1)}, \overline{\varphi}'^{-1}(p)) \to H_k(\overline{\varphi}'^{-1}(p)) \xrightarrow{i_*} H_k(\overline{X}^{(1)}) \to 0$. Thus we can assume $H_k(\widetilde{\varphi}'^{-1}(p); Q) \approx H_k(\widetilde{X}^{(1)}; Q)$. This completes the proof.

By applying Lemma 2.2 for $\partial X^{(1)}$ (if $k \geq 2$) and then for $X^{(1)}$, we complete the proof of Lemma 2.1.

2.3. Proof of Duality Theorem 2. First consider the case that $n \geq 6$ and $m+2 \leq (n-3)/2$. By Lemma 2.1 we have $H_i(\hat{X}^{(2)}, \partial\hat{X}^{(2)} \cup \hat{M}^{(1)}) = 0$ with Z coefficients for $i \leq m+1$ and Q coefficients for $i = m+2$. By using a covering translation of $X^{(1)}$, choose a copy M' of M in $X^{(1)}$ so that $M' \cap M = \emptyset$. M and M' separate $X^{(1)}$ into three parts. Let V be the compact part and N_+, N_- be the others. (See Fig. 1.)

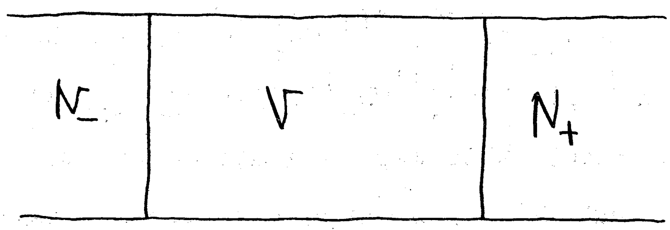

Fig. 1

Note that $X^{(1)}$ can be covered by ascending compact manifolds $V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots$ such that each V_i is separated by two copies of M obtained by covering translations of $X^{(1)}$. For $m+2 \geq 1$ we note that M, V, N_+ and N_- are connected and have the fundamental groups isomorphic to $\pi_1(X^{(1)})$ by inclusions. Since $H_i(\hat{X}^{(2)}, \partial\hat{X}^{(2)} \cup \hat{M}^{(1)}) = 0$, from the Mayer-Vietoris sequence we obtain that

$$\delta : H^i(\hat{X}^{(2)},\partial\hat{X}^{(2)}) \approx H^{i+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)})$$

$$\partial : H_{i+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)}) \approx H_i(\hat{X}^{(2)}, \partial\hat{X}^{(2)})$$

with Z coefficients for $i \leq m$ and Q coefficients for $i = m+1$. Further the coboundary δ is injective for $i = m+1$ with Z coefficients. By excision, we have $H(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)};Z) = H(\hat{V}^{(1)},\partial\hat{V}^{(1)};Z)$, where $H = H^*$ or H_*. Since $H_i(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z)$ is finitely generated abelian for $i \leq m$ and $\dim_Q H_{m+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)};Q) < +\infty$, $H_i(\hat{V}^{(1)},\partial\hat{V}^{(1)};Z)$ is finitely generated abelian for $i \leq m+1$ and $\dim_Q H_{m+2}(\hat{V}^{(1)},\partial\hat{V}^{(1)};Q) < +\infty$. Since $\hat{X}^{(2)}$ is orientable, $\hat{V}^{(1)}$ is orientable. Hence by Duality Theorem 1 we have $\cap\hat{\mu}^{(1)}:H^{i+1}(\hat{V}^{(1)},\partial\hat{V}^{(1)};Z) \approx H_{n-2-i}(\hat{V}^{(1)};Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism, where $\hat{\mu}^{(1)} \in H_{n-1}(\hat{V}^{(1)},\partial\hat{V}^{(1)};Z) = H_{n-1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)};Z)$. We shall show that $\cap\hat{\mu}^{(1)}:H^{i+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)};Z)\approx H_{n-2-i}(\hat{X}^{(2)};Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism. Let V' be such that $V'\supset V$ and V' is a compact manifold separated by two copies of M by covering translations. Further let N'_+, N'_- be two components of $cl(X^{(1)}-V')$ such that $N'_+\subset N_+$ and $N'_-\subset N_-$ and let $A = cl(V'-V)$. Consider the following commutative diagram ($i \leq m$):

$$\begin{array}{ccccc}
H^{i+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N}_+^{(1)}\cup\hat{N}_-^{(1)}) & & \xrightarrow{\approx} & & H^{i+1}(\hat{X}^{(2)},\partial\hat{X}^{(2)}\cup\hat{N'}_+^{(1)}\cup\hat{N'}_-^{(1)}) \\
\downarrow\approx & & & & \downarrow\approx \\
H^{i+1}(\hat{V}^{(1)},\partial\hat{V}^{(1)}) & \xleftarrow{\approx} & H^{i+1}(\hat{V'}^{(1)},\hat{A}^{(1)}\cup\partial\hat{V}^{(1)}) & \rightarrowtail & H^{i+1}(\hat{V'}^{(1)},\partial\hat{V'}^{(1)})
\end{array}$$

$$\begin{array}{ccccc} \Big\downarrow {\scriptstyle \cap\hat{\mu}^{(1)}_{V}} \approx & & \Big\downarrow {\scriptstyle \cap\hat{\mu}^{(1)}_{(V',A)}} & & \Big\downarrow {\scriptstyle \cap\hat{\mu}^{(1)}_{V'}} \approx \\ H_{n-2-i}(\hat{V}^{(1)}) & \xrightarrow{i_*} & H_{n-2-i}(\hat{V}'^{(1)}) & = & H_{n-2-i}(\hat{V}'^{(1)}) \end{array}$$

This shows that the inclusion $i: \hat{V}^{(1)} \subset \hat{V}'^{(1)}$ induces an isomorphism $i_*: H_{n-2-i}(\hat{V}^{(1)};Z) \approx H_{n-2-i}(\hat{V}'^{(1)};Z)$. Using $\varinjlim_{V} H_*(\hat{V}^{(1)};Z) \approx H_*(\hat{X}^{(2)};Z)$, we obtain that the inclusion $j: \hat{V}^{(1)} \subset \hat{X}^{(2)}$ must induce an isomorphism $j_*: H_{n-2-i}(\hat{V}^{(1)};Z) \approx H_{n-2-i}(\hat{X}^{(2)};Z)$. For $i = m+1$ an analogous discussion shows that $j_*: H_{n-3-m}(\hat{V}^{(1)};Z) \to H_{n-3-m}(\hat{X}^{(2)};Z)$ is a monomorphism. [Use additional facts of Lemma 2.1 that $H_m(\hat{N}_{\pm}^{(1)}, \hat{N}_{\pm}'^{(1)};Z) = H_{m-1}(\partial\hat{N}_{\pm}^{(1)}, \partial\hat{N}_{\pm}'^{(1)};Z) = 0$ to prove that $H^{m+2}(\hat{V}'^{(1)}, \hat{A}^{(1)} \cup \partial\hat{V}^{(1)};Z) \to H^{m+2}(\hat{V}'^{(1)}, \partial\hat{V}'^{(1)};Z)$ is injective.] Therefore, combined with $\cap\hat{\mu}^{(1)}$: $H^{i+1}(\hat{V}^{(1)}, \partial\hat{V}^{(1)};Z) \to H_{n-2-i}(\hat{V}^{(1)};Z)$, we have that $\cap\hat{\mu}^{(1)}$: $H^{i+1}(\hat{X}^{(2)}, \partial\hat{X}^{(2)} \cup \hat{N}_{+}^{(1)} \cup \hat{N}_{-}^{(1)};Z) \to H_{n-2-i}(\hat{X}^{(2)};Z)$ is an isomorphism for $i \leq m$ and is a monomorphism for $i = m+1$. Let $\hat{\mu}^{(2)}$ be the image of $\hat{\mu}^{(1)}$ via boundary homomorphism $\partial: H_{n-1}(\hat{X}^{(2)}, \partial\hat{X}^{(2)} \cup \hat{N}_{+}^{(1)} \cup \hat{N}_{-}^{(1)};Z) \to H_{n-2}(\hat{X}^{(2)}, \partial\hat{X}^{(2)};Z)$ (of the Mayer-Vietoris sequence). The composite $H^{i}(\hat{X}^{(2)}, \partial\hat{X}^{(2)};Z) \xrightarrow{\delta} H^{i+1}(\hat{X}^{(2)}, \hat{X}^{(2)} \cup \hat{N}_{+}^{(1)} \cup \hat{N}_{-}^{(1)};Z) \xrightarrow{\cap\hat{\mu}^{(2)}} H_{n-2-i}(\hat{X}^{(2)};Z)$ is given by the map $\cap\hat{\mu}^{(2)}$. Thus we have a desired result for the case that $n \geq 6$ and $m+2 \leq (n-3)/2$. For the case that $n \ngeq 6$ or $m+2 \nleq (n-3)/2$, choose a sufficiently large integer k such that $n+k \geq 6$, $m+2 < (n+k-3)/2$, and consider $S^{(k)} \times \hat{X}^{(2)}$. From the above argument, the map

$$\begin{array}{ccc} \cap\hat{\mu}_{k}^{(2)}: H^{i}(S^{k} \times \hat{X}^{(2)}, S^{k} \times \partial\hat{X}^{(2)};Z) & \to & H_{n+k-2-i}(S^{k} \times \hat{X}^{(2)};Z) \\ \| & & \| \\ H^{0}(S^{k};Z) \times H^{i}(\hat{X}^{(2)}, \partial\hat{X}^{(2)};Z) & & H_{k}(S^{k};Z) \times H_{n-2-i}(\hat{X}^{(2)};Z) \end{array}$$

is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. ($\times$ denotes the cross product.) Since $H_{n+k-2}(S^k \times \hat{X}^{(2)}, S^k \times \partial\hat{X}^{(2)};Z) = H_k(S^k;Z) \times H_{n-2}(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z)$, $\hat{\mu}_k^{(2)}$ can be written as $[S^k] \times \hat{\mu}^{(2)}$, where $[S^k]$ is a fundamental class of S^k and $\hat{\mu}^{(2)} \in H_{n-2}(\hat{X}^{(2)}, \partial\hat{X}^{(2)};Z)$. Using the identity $(1 \times u) \cap ([S^k] \times \hat{\mu}^{(2)}) = [S^k] \times (u \cap \hat{\mu}^{(2)})$ for all $u \in H^i(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z)$, we have an isomorphism

$$\cap\hat{\mu}^{(2)}: H^i(\hat{X}^{(2)},\partial\hat{X}^{(2)};Z) \approx H_{n-2-i}(\hat{X}^{(2)};Z),\ i \leq m$$

and a monomorphism for $i = m+1$. This completes the proof.

For $N \geq 3$ some further restrictions on $X^{(N)}$ are needed.

Duality Theorem N ($N \geq 3$). Suppose $\hat{X}^{(N)}$ is orientable and $X^{(N-1)} \in f(m+N)$ and $\partial X^{(N-1)}$ has at most two components such that each component $B^{(N-1)}$ of $\partial X^{(N-1)}$ is in $f(m+N-1)$ and the inclusion $B^{(N-1)} \subset X^{(N-1)}$ induces an isomorphism $\pi_1(B^{(N-1)}) \approx \pi_1(X^{(N-1)})$. If $H_i(\hat{X}^{(N)};Z)$ and $H_{i-1}(\partial\hat{X}^{(N)};Z)$ are finitely generated abelian for $i \leq m+N-3$ and $\dim_Q H_{m+N-2}(\hat{X}^{(N)};Q) < +\infty$ and $\dim_Q H_{m+N-3}(\partial\hat{X}^{(N)};Q) < +\infty$, then there is a duality $\cap\hat{\mu}^{(N)}: H^i(\hat{X}^{(N)}, \partial\hat{X}^{(N)};Z) \approx H_{n-N-i}(\hat{X}^{(N)};Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism.

Proof. Note that $X^{(N-1)} \in f(m+N)$ implies $X^{(1)} \in f(m+N)$ by using Lemma 1.4. By a similar discussion of 2.3 we can assume that the dimension n of X is sufficiently large so that $n \geq 6$ and $m+N \leq (n-3)/2$. We proceed the proof by induction on N. Let $N = 3$ and first suppose $\partial X = \emptyset$. Since $X^{(1)} \in f(m+3)$, it follows from Lemma 2.1 that $H_i(\overline{M}) \approx H_i(\overline{X}^{(1)})$ with Z coefficients for $i \leq m+2$ and Q coefficients for $i=m+3$. Hence $X^{(2)} \in f(m+3)$ implies $M^{(1)} \in f(m+3)$. Let $X^{(1)} = N_- \cup V \cup N_+$ as in 2.3, where $\partial V = M \cup M'$. Consider the following part of Mayer-Vietoris sequence:

$H_i(\bar{M}^{(1)} \cup \bar{M}'^{(1)}) \to H_i(\bar{N}_+^{(1)} \cup \bar{N}_-^{(1)}) + H_i(\bar{V}^{(1)}) \to H_i(\bar{X}^{(2)})$. Since $M^{(1)}$, $M'^{(1)} \in f(m+3)$ and $X^{(2)} \in f(m+3)$, we have $V^{(1)} \in f(m+3)$. By Lemma 2.1 $H_{i+1}(\hat{V}^{(2)}, \partial\hat{V}^{(2)}) = H_{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}) \overset{\partial}{\approx} H_i(\hat{X}^{(3)})$ and $H^i(\hat{X}^{(3)}) \overset{\delta}{\approx} H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}) = H^{i+1}(\hat{V}^{(2)}, \hat{V}^{(2)})$ with Z coefficients for $i \leq m+1$ and Q coefficients for $i = m+2$. In particular, $H_{i+1}(\hat{V}^{(2)}, \partial\hat{V}^{(2)}; Z)$ is finitely generated abelian for $i \leq m$ and $\dim_Q H_{m+2}(\hat{V}^{(2)}, \partial\hat{V}^{(2)}; Q) < +\infty$. By Duality Theorem 2, $\cap\hat{\mu}^{(2)}: H^{i+1}(\hat{V}^{(2)}, \partial\hat{V}^{(2)}; Z) \to H_{n-3-i}(\hat{V}^{(2)}; Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. According to an analogous method of 2.3, this implies that $\cap\hat{\mu}^{(2)}: H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; Z)$ $\to H_{n-3-i}(\hat{X}^{(3)}; Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Let $\hat{\mu}^{(3)} \in H_{n-3}(\hat{X}^{(3)}; Z)$ be the image of $\hat{\mu}^{(2)}$ by $\partial: H_{n-2}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; Z) \to H_{n-3}(\hat{X}^{(3)}; Z)$. Then the map $\cap\hat{\mu}^{(3)}: H^i(\hat{X}^{(3)}; Z) \overset{\delta}{\approx} H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; Z) \xrightarrow{\cap\hat{\mu}^{(2)}} H_{n-3-i}(\hat{X}^{(3)}; Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Next consider the case that $\partial X \neq \emptyset$. Take the double $D(X^{(2)})$ of $X^{(2)}$. Our assumption implies that $D(X^{(2)}) \in f(m+3)$. [Note that if $\partial X^{(2)}$ is connected, then $\pi_1(D(X^{(2)})) = \pi_1(X^{(2)})$, and if $\partial X^{(2)}$ has two components, then $\pi_1(D(X^{(2)})) = \pi_1(X^{(2)}) \times Z$.] Using that $H_i(D(\hat{X}^{(3)}); Z)$ is finitely generated abelian for $i \leq m$ and $\dim_Q H_{m+1}(D(\hat{X}^{(3)}; Q) < +\infty$, from the case of manifolds with empty **boundary** $\cap\hat{\mu}_D^{(3)}: H^i(D(\hat{X}^{(3)}); Z) \to H_{n-3-i}(D(\hat{X}^{(3)}); Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Note that $\hat{X}^{(3)}$ is a retract of $D(\hat{X}^{(3)})$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(\hat{X}^{(3)},\partial\hat{X}^{(3)}) \arrow[r] \arrow[d, "\cap\hat{\mu}^{(3)}"] & H^i(D(\hat{X}^{(3)})) \arrow[r] \arrow[d, "\cap\hat{\mu}_D^{(3)}"] & H^i(\hat{X}'^{(3)}) \arrow[r] \arrow[d, "\cap\hat{\mu}'^{(3)}"] & 0 \\
0 \arrow[r] & H_{n-3-i}(\hat{X}^{(3)}) \arrow[r] & H_{n-3-i}(D(\hat{X}^{(3)})) \arrow[r] & H_{n-3-i}(\hat{X}'^{(3)},\partial\hat{X}'^{(3)}) \arrow[r] & 0
\end{tikzcd}$$

, where $\hat{X}'^{(3)}$ is another copy of $\hat{X}^{(3)}$ in $D(\hat{X}^{(3)})$ and the top and bottom sequences are exact. Let $i \leq m$. The middle vertical map $\cap\hat{\mu}_D^{(3)}$ is an isomorphism, and hence the right vertical map is surjective and the left vertical map is injective. (Let $\hat{\mu}^{(3)}$ be the image of $\hat{\mu}_D^{(3)}$ by $H_{n-3}(D(\hat{X}^{(3)});Z) \to H_{n-3}(D(\hat{X}^{(3)}),\hat{X}'^{(3)};Z) = H_{n-3}(\hat{X}^{(3)},\partial\hat{X}^{(3)};Z)$.) Also, consider the following commutative diagram:

$$\begin{tikzcd}
H^{i-1}(\partial\hat{X}^{(3)}) \arrow[r, "\delta"] \arrow[d, "\cap\partial\hat{\mu}^{(3)}"] & H^i(\hat{X}^{(3)},\partial\hat{X}^{(3)}) \arrow[r, "j^*"] \arrow[d, "\cap\hat{\mu}^{(3)}"] & H^i(\hat{X}^{(3)}) \arrow[r, "i^*"] \arrow[d, "\cap\hat{\mu}^{(3)}"] & H^i(\partial\hat{X}^{(3)}) \arrow[d, "\cap\partial\hat{\mu}^{(3)}"] \\
H_{n-3-i}(\partial\hat{X}^{(3)}) \arrow[r, "i_*"] & H_{n-3-i}(\hat{X}^{(3)}) \arrow[r, "j_*"] & H_{n-3-i}(\hat{X}^{(3)},\partial\hat{X}^{(3)}) \arrow[r, "\partial"] & H_{n-4-i}(\partial\hat{X}^{(3)})
\end{tikzcd}$$

, where the top and bottom sequences are exact and the map $\cap\partial\hat{\mu}^{(3)}:H^{i-1}(\partial\hat{X}^{(3)};Z) \to H_{n-3-i}(\partial\hat{X}^{(3)};Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$ by the case of manifolds with empty boundary. Since $\cap\hat{\mu}^{(3)}:H^i(\hat{X}^{(3)},\partial\hat{X}^{(3)};Z) \to H_{n-3-i}(\hat{X}^{(3)};Z)$ is a monomorphism for $i \leq m$, it follows that $\cap\hat{\mu}^{(3)}:H^i(\hat{X}^{(3)};Z) \to H_{n-3-i}(\hat{X}^{(3)},\partial\hat{X}^{(3)};Z)$ is a monomorphism and hence an isomorphism for $i \leq m$. By the five lemma, this implies that $\cap\hat{\mu}^{(3)}:H^i(\hat{X}^{(3)},\partial\hat{X}^{(3)};Z) \to H_{n-3-i}(\hat{X}^{(3)};Z)$ is an isomorphism for $i \leq m$. This map is also a monomorphism for $i = m+1$, since $\cap\hat{\mu}_D^{(3)}:H^{m+1}(D(\hat{X}^{(3)});Z) \to H_{n-3-(m+1)}(D(\hat{X}^{(3)});Z)$ is injective. This completes the proof of

$N = 3$.

By assuming Duality Theorem N-1, we must show Duality Theorem N. However, the proof is quite parallel to the proof of $N = 3$ and left to the reader. This completes the proof.

3. An application

Consider a closed, connected n-manifold X^n satisfying the following (1) and (2):

(1) $\pi_1(X)$ admits a tower of subgroup $\pi_1(X) = G_0 > G_1 > \dots > G_r = \{1\}$ such that for each i, $i=1,2,\dots,r$, G_{i-1}/G_i has a finite index or is an infinite cyclic group,

(2) $\pi_i(X)$, $2 \leq i \leq (n-1)/2$, is finitely generated abelian and, when n is even, $\dim_Q \pi_{n/2}(X) \otimes Q < +\infty$.

The class of such manifolds X is denoted by $\mathfrak{M}$.

3.1 <u>Definition</u>. Let $X \in \mathfrak{M}$. $\pi_1(X)$ is said to have <u>rank</u> R, if infinite cyclic quotient groups occur at R times in a tower of $\pi_1(X)$.

We must prove

3.2 <u>Lemma</u>. R <u>does not depend on a choice of towers of</u> $\pi_1(X)$.

3.3 <u>Definition</u>. $\rho = \rho(X) = n - R$ is called the (<u>topological</u>) <u>Kodaira dimension</u> of $X^n \in \mathfrak{M}$.

We shall show the following:

3.4 <u>Theorem</u>. ρ <u>is a non-negative integer except for one</u>.

3.5. <u>Proof of Lemma 3.2 and Theorem 3.4</u>. First note that $Z[\pi_1(X)]$ is Noetherian by Corollary 1.5. We apply Duality Theorem N ($N \geq 0$). Let n be even, say, $n = 2n'$. For all $i \leq n'-R$ we have $H^i(\overline{X};Z) \approx H_{n-R-i}(\overline{X};Z)$. When $i \leq n'-R$, $n-R-i \geq n-R-(n'-R) = n - n' = n'$. Thus, $H_i(\overline{X};Z)$ is finitely generated abelian for $i \geq n'$, hence for all i. Let n' be odd, say, $n = 2n'+1$. For all $i \leq n'-R$, we have $H^i(\overline{X};Z) \approx H_{n-R-i}(\overline{X};Z)$. When $i \leq n'-R$, we have $n-R-i \geq n-R-(n'-R) = n-n' = n'+1$. Thus, $H_i(\overline{X};Z)$ is finitely generated abelian for $i \geq n'+1$, hence for all i. As a result, for each $n \geq 1$ there is a duality $H^i(\overline{X};Z) \approx H_{n-R-i}(\overline{X};Z)$ for all i. This implies that R does not depend on any choice of towers of $\pi_1(X)$. By taking $i = 0$, $\rho = n-R \geq 0$. Since $\overline{X}$ is simply connected, $\rho \neq 1$. This completes the proof.

We further state individually results for X in $\mathfrak{M}$ on each Kodaira dimension ρ.

3.6. $\rho = 0$. In this case, $H_*(\overline{X};Z) = 0$, hence $\overline{X}$ is contractible, and X is $K(\pi,1)$. In particular, $\pi_1(X)$ is torsion-free. If $\pi_1(X)$ is abelian and $n \geq 5$, X is topologically homeomorphic to an n-torus $S^1 \times \cdots \times S^1$. More generally, if $\pi_1(X)$ is a poly-infinite-cyclic group and $n \geq 5$, then according to C.T.C.Wall[14] $\overline{X}$ is homeomorphic to $\mathbb{R}^n$ and for any closed n-manifold X' with $\pi_1(X') \approx \pi_1(X)$, any homotopy equivalence $X' \to X$ is homotopic to a topological homeomorphism.

3.7. $\rho = 2$. Then $H_*(\bar{X};Z) \approx H_*(S^2;Z)$. Hence $\bar{X}$ is is homotopy equivalent to S^2. Assume $\pi_1(X)$ admits a special tower $\pi_1(X) = G_0 \supset G_1 \supset \cdots \supset G_R \supset \{1\}$ such that for each i, $i=1,2,\ldots,R$, G_{i-1}/G_i is an infinite cyclic group and G_R is a finite group. Then G_R is $\{1\}$ or Z_2. To see this, consider the cover $X^{(R)}$ corresponding to G_R. First, suppose $X^{(R)}$ is orientable. If $G_R \neq \{1\}$, then let Z_p be a cyclic subgroup of G_R and $\hat{X}^{(R)}$ be the corresponding cover. By Duality Theorem R, there is a duality $0=H^1(\hat{X}^{(R)};Z) \approx H_1(\hat{X}^{(R)};Z)=Z_p$, which is a contradiction.[Note that $H_*(\hat{X}^{(R)};Z)$ is finitely generated abelian, since $X^{(R)}$ is homotopy equivalent to a complex of finite type(,i.e., skelton-finite complex).] Hence $G_R = \{1\}$. If $X^{(R)}$ is non-orientable, by the orientable case, the orientation cover of $X^{(R)}$ is simply connected. This implies $G_R = Z_2$. As a simple consequence, if $\pi_1(X)$ is abelian, then $\pi_1(X)$ is isomorphic to Z^{n-2} or $Z^{n-2}+Z_2$.

3.8. $\rho = 3$. Since $H_*(\bar{X};Z) \approx H_*(S^3;Z)$, X is homotopy equivalent to S^3. Assume $\pi_1(X)$ admits a special tower $\pi_1(X) = G_0 \supset G_1 \supset \cdots \supset G_R \supset \{1\}$ as in 3.7. We shall show that every abelian subgroup of G_R is cyclic. Hence G_R has a period > 1. In particular, if $\pi_1(X)$ is abelian, then $\pi_1(X) \approx Z^{n-3}+Z_m$ $(m \geq 1)$. To see this, first assume $X^{(R)}$ is orientable. Let A be an abelian subgroup of G_R. Let $\hat{X}^{(R)}$ be the corresponding cover. By Duality Theorem R, we have $H^i(\hat{X}^{(R)};Z) \approx H_{3-i}(\hat{X}^{(R)};Z)$ for all i. Since $H_1(\hat{X}^{(R)};Z)$ is a finite group, $H_2(\hat{X}^{(R)};Z)= 0$. This implies $H_2(A;Z) = 0$; so, A is cyclic. [Use a cyclic decomposition of A.] Now we must show that $X^{(R)}$ is necessarily orientable. From a successive use of the Novikov-Siebenmann splitting theorem (See L.C.Siebenmenn[10].), we have $S^3 \times X^{(R)} \cong M^6 \times R^{(R)}$ for a closed 6-manifold M^6. [Note that the Wall-Siebenmann obstruction $\sigma(S^3 \times X^{(i)})$, i=1,2,..,R, in the reduced projective class group $\tilde{K}_0(Z[\pi_1(S^3 \times X^{(i)})]$ vanishes, since the Euler characteristic $\chi(S^3)$ of S^3 is 0.] Hence $H_i(S^3 \times X^{(R)};Z) = 0$, $i \geq 7$. $X^{(R)}$ is non-orientable if and only if $H_6(S^3 \times X^{(R)};Z)= 0$. In other words, $H_i(X^{(R)};Z) = 0$, $i \geq 4$, and $X^{(R)}$ is non-orientable if and only if $H_3(X^{(R)};Z)=0$. Note that there is a duality $\cap\bar{\mu} : H^i(S^3 \times X^{(R)};Z_2) \approx H_{6-i}(S^3 \times X^{(R)};Z_2)$ for all i. As an analogy of a fact shown in 2.3, this duality can be interpretted as $\cap\mu : H^i(X^{(R)};Z_2) \approx H_{3-i}(X^{(R)};Z_2)$ for all i. This shows that the Euler characteristic $\chi(X^{(R)};Z_2)$ over the **coefficient field** Z_2 is 0. We need the following lemma:

3.8.1 Lemma. Suppose a space K has a finitely generated integral homology group $H_*(K;Z)$. Then the Euler characteristic of K is independent of a coefficient field which is used.

From this lemma, the usual Euler characteristic(, that is, the Euler characteristic over Q) $\chi(X^{(R)}) = 0$. Suppose $X^{(R)}$ is non-orientable. We count the Betti numbers of $H_*(X^{(R)};Z)$. We have $0 = \chi(X^{(R)}) = \beta_0(X^{(R)}) - \beta_1(X^{(R)}) + \beta_2(X^{(R)}) = 1-0+ \beta_2(X^{(R)})$

≥ 1, which is a contradiction.[Note that $H_1(X^{(R)};Z)$ is a finite group. **Therefore, $X^{(R)}$ is orientable.**

3.8.2. Proof of Lemma 3.8.1. Since $H_*(K;Z)$ is finitely generated, by the proof of E.H.Spanier[11],Lemma 9, p 246, there is a finitely generated free chain complex C chain equivalent to the free geometric chain complex $C(K;Z)$. Then the assertion follows from the Euler-Poincaré formula. In fact, for a field F

$$\begin{aligned}\chi(K;F) &= \textstyle\sum_i (-1)^i \dim_F H_i(K;F)\\ &= \textstyle\sum_i (-1)^i \dim_F H_i(C;F)\\ &= \textstyle\sum_i (-1)^i \dim_F C_i \otimes F\end{aligned}$$

and $\dim_F C_i \otimes F$ is independent of a choice of fields F. This completes the proof.

3.9. $\rho = 4$. In this case $\bar{X}$ is only a simply connected Poincaré 4-complex. $\bar{X}$ is homotopy equivalent to the adjunction space of **a 4-cell B^4 to a bouquet** $S^2 \vee \ldots \vee S^2$ by a map $a : \partial B^4 \to S^2 \vee \ldots \vee S^2$. From this, one can see that the homotopy type of $\bar{X}$ is characterized by the symmetric inner product $H^2(\bar{X};Z) \times H^2(\bar{X};Z) \xrightarrow{\cup} H^4(\bar{X};Z)=Z$. (cf. J.W.Milnor-D.Husemoller[7].)

3.10. $\rho \geq 5$. $\bar{X}$ is a simply connected Poincaré ρ-complex. Let precisely $\pi_1(X) = G_0 \supset \hat{G}_0 \supset G_1 \supset \hat{G}_1 \supset \ldots \supset G_R \supset \hat{G}_R = \{1\}$, where for each i, $i=0,1,..,R$ $G_i/\hat{G}_i$ has a finite index and for each i, $i=1,2,..,R$, $\hat{G}_{i-1}/G_i$ is an infinite cyclic group. If the Wall-Siebenmann obstruction $\sigma(X^{(i)})$ in $\tilde{K}_0(Z[G_i])$ vanishes, then $\bar{X}$ is piecewise-linearly homeomorphic to the splitting $M \times \mathbb{R}^R$ for a piecewise-linear closed manifold M by the Novikov-

Siebenmann splitting theorem. In particular, if G contains a poly-infinite-cyclic group (of rank R) as a subgroup with finite index (for example, G is abelian), then $\bar{X}$ splits: $\bar{X} \cong M\times\mathbf{R}^R$, because $\tilde{K}_0(Z[P]) = 0$ for all poly-infinite-cyclic groups P of finite rank. (cf. W.C.Hsiang [3].)

We consider low-dimensional consequences. For example, closed 3-manifolds with abelian fundamental groups are contained in $\mathfrak{M}$. It follows that the possible group as the fundamental groups is Z^3, $Z+Z_2$, Z or Z_m. This is also a classical result due to K. Reidemeister[9]. Certainly, the universal cover is contractible (more precisely, R^3 modulo Poincaré conjecture) or has the homotopy type of S^2 or S^3 according as the Kodaira dimension $\rho = 0$ or 2 or 3. Next, for example, consider a closed 4-manifold M with $\pi_1(M) = Z^r$, $r \geq 5$. [Note that such a manifold M does exist.] As a simple consequence of Theorem 3.4, $\pi_2(M)\otimes Q$ is necessarily infinitely generated over Q.

4. Further discussions

4.1. Although we established the duality theorem with integral coefficients, under the same hypotheses we can have every torsion-free group as a coefficient of the duality. This fact is based on A.Kawauchi[4]. Further, from [4], if our class $f(k)$, $k\geq 1$, is replaced by Wall's class NFk in [13] and $\dim_Q H_k(\hat{X}^{(N)},\partial\hat{X}^{(N)};Q) < +\infty$(or $\dim_Q H_k(\hat{X}^{(N)};Q) < +\infty$ or $\dim_Q H_k($

$\partial\hat{X}^{(N)};Q) <+\infty)$ is replaced by a condition that $H_k(\hat{X}^{(N)}, \partial\hat{X}^{(N)};Z)$ (or $H_k(\hat{X}^{(N)};Z)$ or $H_k(\partial\hat{X}^{(N)};Z)$) is finitely generated abelian , then Duality Theorem $N(\geq 1)$ holds for an arbitrary coefficient group.

4.2. One can obtain a duality of a type in Kawauchi[5] for iterated infinite cyclic coverings under rather simple hypotheses, but details remain open.

4.3. In this paper, we worked in the piecewise-linear category. However, a manofold may be a topological manifold , because given m, it suffices to establish an i-duality, $i \leq m$, of a manifold with a sufficiently large dimension n in contrast with m, and a transverse-regularity of topological manifolds in high-codimension and a surgery on low-dimensional handles in high-dimensional topological manifolds can be done just as piecewise-linear manifolds.

Department of Mathematics,Osaka City University,Osaka ,Japan

References

1. D.B.A.Epstein; Ends, Topology of 3-Manifolds and Related Topics, M.K.Fort, Jr., Prentice-Hall, 1962, 110-117.

2. M.A.Gutiérrez; An exact sequence calculation for the second homotopy of a knot, Proc. A.M.S. 32(1972), 571-577.

3. W.C.Hsiang; A splitting theorem and the Künneth formula in algebraic K-theory, Algebraic K-theory and its Geometric Applications, Lecture Notes No.108, Springer-Verlag, 1969, 72-77.

4. A.Kawauchi; A partial Poincaré duality theorem for infinite cyclic coverings, Quart.J.Math. 26(1975), 565-581.

5. A.Kawauchi; On quadratic forms of 3-manifolds, Invent. math. (to appear)

6. J.W.Milnor; Infinite cyclic coverings, Conference on the Topology of Manifolds, Prindle, Weber and Schmidt, Boston Mass., 1968, 115-133.

7. J.W.Milnor-D.Husemoller; Symmetric Bilinear Forms, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

8. L.P.Neuwirth; Knot Groups, Princeton Univ. Press, Princeton, 1968

9. K.Reidemeister; Kommutative Fundamentalgruppen, Monatsh.f.Math. und Physik, 43(1936), 20-28.

10. L.C.Siebenmann; The obstruction to finding a boundary for an open manifold of dimension greater than five, Thesis, Princeton Univ., 1965.

11. E.H.Spanier; Algebraic Topology, McGraw-Hill, 1966.

12. J.Stallings; On fibering certain 3-manifolds, Topology of 3-

Manifolds and Related Topics,M,K.Fort,Jr.,Prentice-Hall, 1962,95-100.

13.C.T.C.Wall;Finiteness conditions for CW complexes, Ann. of Math. 81(1965),56-69.

14.C.T.C.Wall;The topological space-form problems,Proceedings of Georgia Conference in Topology,J.C.Cantrell and C.H. Edwards,Jr., 1969,319-331.